DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

78. Proposed by J. MARCUS BOORMAN, Consultative Mechanician and Counselor at Law, Woodmere, Long Island, N. Y.

Solve $x^2+xy=10\ldots\ldots(1)$; $y^2+xy=15\ldots\ldots(2)$, for all the roots, and prove that they are the roots.

I. Summary of Solutions by J. OWEN MAHONEY, M. Sc., Lynnville, Tenn.; F. R. HONEY, Ph. B., Instructor at Trinity College, New Haven, Conn.; J. SCHEFFER, A. M., Hagerstown, Md.; G. B. M. ZERR, A. M., Ph. D., The Russell College, Lebanon, Va.; P. S. BERG, A. M., Principal of Schools, Larimore, N. D.; A. H. BELL, Hillsboro, Ill.; H. C. WILKES, Skull Run, W. Va.; CHARLES C. CROSS, Laytonsville, Md.; I. H. BRYANT, A. M., Ft. Smith High School, Ft. Smith, Ark.; and COOPER D. SCHMITT, A. M., University of Tennessee, Knoxville, Tenn.

Divide (1) by (2), then $x/y=\frac{2}{3}\ldots\ldots$ (3) ; x and y must be plus or minus together. Then x from (3) in (1), gives $y^2=9$, or $y=\pm3$. Also, $x(=2y/3)=\pm2$. $\therefore$ $x=\pm2$, $y=\pm3$ are the roots, which can be proved by direct substitution. [MAHONEY, HONEY, SCHEFFER, AND BELL.]

Adding (1) and (2), and extracting square root, $x+y=\pm5\ldots\ldots(4)$. Then (3) in (4) gives, $y=\pm3$, $x=\pm2$. [ZERR.]

Subtracting (2) from (1), $x^2-y^2=-5\ldots\ldots(5)$. Then (5) by (4), $x-y=\mp1\ldots\ldots(6)$. From (4) and (6), $x=\pm2$, $y=\pm3$. [BERG, AND WILKES.]

Solve (1) for x, then $x=[-y\pm\sqrt{(40+y^2)}]/2$ $\ldots\ldots(7)$. Substituting (7) in (2), and reducing, $y=\pm3$. Similarly, $x=\pm2$. [CROSS.]

Substitute vy for x and solve ; then $v=\frac{2}{3}$ or -1. By substituting and reducing, $x=\pm2$, and $y=\pm3$. Then if $v=\frac{2}{3}$, $x=\pm2$, $y=\pm3$. If $v=-1$, $x^2=\infty$, $x=\pm\infty$, $y=\mp\infty$. The first values satisfy the equations. Substitute the second values, $x^2=\infty$, $x=+\infty$, $y=-\infty$, $xy=-\infty$, $x^2+xy=\infty-\infty$. Since $\infty-\infty$ is an indeterminate expression, it may equal any numbers. Therefore the equations are satisfied for the last values of x and y as well as for the first values. [BRYANT.]

From (1), $y=(10-x^2)/x$. In (2), we get $10-x^2+(100-20x^2+x^4)/x^2=15$. Whence $25x^2-100=0\ldots\ldots(8)$; or $x=\pm2$. Being an equation of the fourth degree we ought to have four answers. We can write (8), $0x^4+0x^3+25x^2-100=0$. Since coefficients of two highest powers are zero, this indicates two infinity roots, which may be *claimed* defective in solutions in print. [SCHMITT.]

See *Analyst*, Vol. VIII, page 111, and Vol. IX, page 53, as well as solution below, for discussion as to whether $x=\pm\infty$ and $y=\mp\infty$ are also roots of the equations. [EDITOR.]

II. Solution by the PROPOSER.

The equations are fourth degree in x ; y ; the *singular* case in $x^2+xy=a$ $\ldots\ldots(A)$; $y^2+exy=A+d\ldots\ldots$ (B). *sub-ultimate* fourth degree when the positive are the negative roots reversed in signs. Here $a=10$; $A=15$; e (*variable*) $=1$; $d=F(e-1)=0$.

Transpose (I), $x^2=10-xy$; then transpose (II), and multiply, giving $x^2y^2=150-25xy+x^2y^2$......(III). $\therefore$ $xy=6$......(IV), and $(1-1)x^2y^2-25xy+(12\frac{1}{2})^2[1/(1-1)]=(12\frac{1}{2})^2/(1-1)-150$......(III$a$). By (IV) and (I), (II); $x^2=4$; $y^2=9$; $\therefore$ $x=\pm2$; to $y=\pm3$; four *true* roots......(V). But x^2y^2 (vanished) is quadratic; it has therefore a *second* root $-xy=-6$; hence in (I), $x^2-(-6)=10$; *i. e.* $x^2=10+(-6)$......(VI). $\therefore$ extract negatively $x=\mp2$; $y=\mp3$ [by parity in (II)]; and $x=\pm2$; $y=\pm3$; ......(V) are the eight required roots of Case (I), (II) and are *all* its roots. Q. V. D.

Presumably (*visum*) demonstrated.

PROOF. Case (I) (II) is *biquadrate* because $xy=6$ that yields *two* pairs of roots flows *equally from* both only possible extractions of the *quadrate* x^2y^2...... $\therefore$ $-xy=6$ *must* yield, failing other derivatives of (III), also the second set of roots. For now obviously (VI) cannot be $x^2=16$ to $y^2=21$ (nor $y^2=9$); because then (I) a whole number $16+4\sqrt{(21)}=10$; or an irrational surd=a rational number, *i. e.* reducing $\sqrt{(21)}=\mp\frac{3}{2}$, which is *not* true. Nor can $x=\int\sqrt{(-1)}$; if so therefore $y=\int_{|}[-\sqrt{(-1)}]$; to make xy positive). $\therefore$ $xy-\int^2 \mathrm{I}=10$.... is (I); $xy-\int_{|}^{2}\mathrm{I}=15$ is (II) which *cannot* be for 10; 15; both positive. If it can be then ..... (I); (II) become: (I) $xy-x^2=10$ and (II) $xy-y^2=15$; then must $\mp_{|}y$; to $\pm_{|}x$; be true. Deduct (I) from (II), $\therefore$ $x^2-y^2=5=st$....(VIII). Let $x+y=s$; $\therefore$ $x-y=t$; add, $\therefore$ $x=\frac{1}{2}(s+t)$; $y=\frac{1}{2}(s-t)$; $\therefore$ $xy=\frac{1}{4}(s^2-t^2)$. Multiply by 4, change signs, etc., $\therefore$ $x^2=\frac{1}{4}(s^2+2st+t^2)$ put into (I), $s^2+2st+t^2-4xy=-40$ ....(IX), [$\pm$ are changed by $\sqrt{(-1)}$]; and $y^2=\frac{1}{4}(s^2-2st-t^2)$ put into (II), $s^2-2st+t^2-4xy=-60$. Add the 2 first, $\therefore$ $x^2+y^2=\frac{1}{2}(s^2+t^2)$, replace in (I); then $2x^2-4xy+2y^2+2st=-40$; but (VIII), $st=5$; $\therefore$ $(x-y)^2=-20-5=-25$; *i. e.* $x-y=\pm5\sqrt{(-1)}$......(X), and by (VIII) $x+y=\mp\sqrt{(-1)}$ .... (XI); add (X), (XI), $\therefore$ $2x=\pm4\sqrt{(-1)}$; $x=\pm2\sqrt{(-1)}$; and by (XI)$-$(X) $y=\mp3\sqrt{(-1)}$; *which do not* satisfy (I), (II). $\therefore$ x; y are not imaginary roots of (I) (II) and equation (IX) is falsely put as equation (I). But correct the false factors $\sqrt{(-1)}$ in (IX), (X), (XI) and they by same process yield the roots of (I) (II) found.

SECOND (see note).

(I) (II) have *no sort* of unreal roots. If they may $x={}_{|}a+ei$; $y={}_{|}b+ci$; $\therefore$ $x^2=a^2-e^2+2_{|}aei$ and $y^2=b^2-c^2+2_{|}bci$. $\therefore$ $ab-ec+(ac+be)i=xy$....(N), and $b^2-a^2+e^2-c^2=5=(\mathrm{II})-(\mathrm{I})$......($P$); because to cancel i (I) (II) we *must have*, $2aei=(ac+be)i=2bci$......(Q). For *else* rationals 10; 15; (a^2-e^2), etc., *have to* equal $f\sqrt{(-1)}$ or $f_{|}\sqrt{(-1)}$; etc., *i. e.* real numbers *can be* partly *un*-real, which is absurd! $\therefore$ by (Q) $2a=b+ac/e$; and $2b=a+be/c$...... [These easily reduce to $c=\int i$; $e=\int_{|} i$...... multiplied by i. $\therefore$ $ci=fii$; a *real* number. So our postulate that *ci can be un*-real is false.] $\therefore$ $2b-a=be/c$ and $2a-b=ac/e$. $\therefore$ $b(2-e^2/ec)=a$ and $a(2-c^2/ec)=b$. [Thence $b<a$ and $b>a$! if $c=e$, as here

proven.] Multiply $\therefore ab[4-2(c^2+e^2)/ec+1]=ab$; cancel $ab=ab$, reduce, etc., $\therefore 2ce=c^2+e^2$, $\therefore (c-e)^2=0$......(K). $\therefore c=e$; ...... $\therefore (P_{|})$ is $b^2-a^2=5$ and our assumed *un*-real ci ; ei ; *do not exist*. $\therefore x=a_{|}$; $y=b_{|}$ *real* numbers in (N) (Q) just as we found above.

Again, (IIIa) reduced is $\sqrt{(1-1)}xy=[\pm 1/\sqrt{(1-1)}]\{[12\frac{1}{2}\pm\sqrt{[156\frac{1}{4}-150(1-1)]}\}$......(XII). $\therefore xy=[\pm 1/(1-1)](12\frac{1}{2}\pm 2\frac{1}{2})$. $\therefore x_2y_2=\pm 15/(1-1)$; $x_ay_a=\pm 10/(1-1)$......(XIII). [Unless in $\int 1/(1-1)$ of (XII); $-150(1-1)=0$, for which value $xy=[1/(1-1)](12\frac{1}{2}\pm 12\frac{1}{2})=x_3y_3=25/(1-1)$, or $x_4y_4=12\frac{1}{2}[(1-1)/(1-1)$......(XIV)]. Hence by (XII) and (I), (II) $x_a{}^2=\pm 10[1-1/(1-1)]$; $y_a=\pm 10[\frac{3}{2}-1/(1-1)]$; $x_2{}^2=\pm 10[1-1.5/(1-1)]$; $y_2{}^2=\pm 15[1-1/(1-1)]$; multiply x_2 ; y_2 ; $\therefore x_2y_2=\sqrt{\{15[10-10/(1-1)]\}}$; (or ?) .....(XV) ; but (XV) *is not* equation (XIII) nor does it nor (XIV) any way satisfy (I) (II). $\therefore x_a, y_a, x_2, y_2$, are *not*, either as by (XIII) or (XIV) roots of (I) (II). Q. E. D.

Last, put $r=$ ratio $y : x$. $\therefore y=rx$; $x^2(1+r)=10$......(Ia) ; $x^2(r^2+r)=15$ .....(H). $\therefore r=\frac{3}{2}$; $r_{|}=-1$......(K). Ratio $\frac{3}{2}$ gives $x=\pm 2$ to $y=\pm 3$...... (V), or $x=\mp 2$; $y=\mp 3$......(VI) above. But by $r_{|}=-1$; (Ia) gives $x^2(1-1)=10$; $x_5=\pm_{|}\sqrt{[10/(1-1)]}$; and ($H$) $x_0=\pm_{|}\sqrt{[15/(1-1)]}$; $\therefore y_5=\pm_{|}\sqrt{[10/(1-1)]}$; $y_0=\mp_{|}\sqrt{[15/(1-1)]}$; ......(M). Whether or no (M) be the quasi roots of (XIV) *none* satisfy (I) (II) ! Besides they are *too many* and give (I)(II) 10, 14 or 18 roots ! So quasi-results (XV) (M) *are not* roots and ratio $r_{|}=-1$ yields *no* root. $\therefore$ ratio $\frac{3}{2}$ covers all eight roots of (I) (II), viz. roots (V) direct, or (V) with contrary signs as above. Q. V. D.

Finally, solve (I) (II) *generally*. $\therefore x=\pm a/\sqrt{(a+A)}$; $y=\pm A/\sqrt{(a+A)}$; $xy=aA/(A+a)$. Now $d=(e-1)[(aA)/(a+A)]$, so equation (A) is equation (I), but (B) is $y^2+exy=A+[aA/(a+A)](e-1)$......($L$) ; hence $x^2y^2=Aa+[a^2A(e-1)/(a+A)]-(ae+A)xy+ex^2y^2$ ... is (IIIa) *by generalizing*, and therefore $x^2y^2-[a+(A+1)/(e-1)]xy+a^2A/(a+A)+aA/(e-1)=0$ is the general form ......(J). $\therefore$ take (A) $x^2=a-xy$ and (B) $y^2=A+d-exy$; and let $e=2$. $\therefore d=(e-1)xy=xy=6$; $A=15$; $y^2+2xy=21$......($B_{|}$) ; $(2-1)x^2y^2-41xy+210=0$......(C). $\therefore (xy-(20\frac{1}{2})^2=210\frac{1}{4}$......($C_{|}$). $\therefore xy=20\frac{1}{2}\mp 14\frac{1}{2}$, *i. e.* $xy=6$ ......(IV), same as by (I) (II).

But also $x_{||}y_{||}=20\frac{1}{2}+14\frac{1}{2}=35$......($D$). $\therefore$ by (A) or (I) $x_{||}{}^2=10-35=-25$; *while* by ($B_{|}$) $y^2=21-70=-49$. $y_{||}=\mp_{|}7\sqrt{(-1)}$; $x=\pm_{|}5\sqrt{(-1)}$.... (E) roots that *do* satisfy (I) or (A) ; ($B_{|}$) *as well as* $x=\pm_{|}2$; $y=\pm_{|}3$......(V), —and a general sub-ultimate fourth degree (A) ($B_{|}$) has, its eight roots in pairs "where the positive are the negative roots *reversed in signs*."

Half are roots *both of* (A) ($B_{|}$) and of (I) (II) because (I) is *identically* equation (A). But changing ($B_{|}$) in (II) changes its *second sets* of roots,—to say it *destroys* them is absurd! So therefore (I) (II) have *their own half set* of co-roots besides the roots that (II) shares also with ($B_{|}$) because (I) is *identically* (A). Q. E. D.

As the second set *cannot be unreal*, they *are real* and derive of x^2y^2 *vanished* by *symmetry* which is *why* (IIIa) *cannot* diverge its roots (VI) from (V) as

equation (C) *does* diverge its roots (E) from the roots (V). Therefore, *no other result being possible*, the roots that (II) has with (I) but not with ($B_{|}$) are $x_{|}=\mp_{|}2$; $y_{|}=\mp_{|}3$ and these with $x=\pm_{|}2$; $y=\pm_{|}3$ are *the eight*, and are *all* the roots of (I) (II) a fourth degree *singular* case. Q. E. D.

NOTE.—The $\pm_{|}\ldots\ldots\pm_{|}$; $\pm_{|}\ldots\ldots\mp_{|}$; or $(_{|})\ldots\ldots(_{|})$; mean "change signs in *unison only*." Thus in "SECOND" $_{|}a\ldots\ldots{}_{|}b$ (*not a* prime, etc.)

GEOMETRY.

81. Proposed by CHAS. C. CROSS, Laytonsville, Md.

A circle is drawn bisecting the lines joining the points of contact of the escribed circles with the sides produced. Another circle is drawn passing through the centers of the circles drawn tangent externally to the in-circle and internally to the sides of the triangle. Prove that the centers of these two circles, the in-center and the circumcircle are collinear.

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Let ABC be the triangle; O_a, O_b, and O_c the centers of the escribed circles tangent externally to the sides a, b, and c respectively, h and g the points of tangency of circle whose center is O_a with the sides c and b produced; e and f the points of tangency of the circle, center O_b, with the sides c and a produced; d and k the points of tangency of the circle, center O_c, with the sides b and a; O the center of the in-circle; a, b, and c the centers of the circles described tangent to circle, center O, and the sides b and c, c and a, and a and b; E, G, and K the feet of the perpendiculars from the centers a, b, and c to the side a; P, Q, and R the middle points of the lines de, gh, and hk respectively; F, L, and D the feet of the perpendiculars let fall from P, Q, and R on the side a; and O', M', M the centers of the circles through a, b, and c, A, B, and C, and P, Q, and R respectively.

Lemma—The coördinates of the center of a circle passing through (x_1, y_1), (x_2, y_2), (x_3, y_3), are given by

$$\alpha=\frac{(x_1^2+y_1^2)(y_3-y_2)+(x_2^2+y_2^2)(y_1-y_3)+(x_3^2+y_3^2)(y_2-y_1)}{2x_1(y_3-y_2)+2x_2(y_1-y_3)+2x_3(y_2-y_1)}\ldots\ldots\ldots\ldots(1).$$

$$\beta=\frac{(x_1^2+y_1^2)(x_3-x_2)+(x_2^2+y_2^2)(x_1-x_3)+(x_3^2+y_3^2)(x_2-x_1)}{2y_1(x_3-x_2)+2y_2(x_1-x_3)+2y_3(x_2-x_1)}\ldots\ldots\ldots\ldots(2).$$

$Bk=Cf=s-a$, $Cg=Ad=s-b$, $Bh=Ae=s-c$.

Taking B as origin and axes rectangular we get coördinates of M, $\{\frac{1}{2}a, \frac{1}{2}a\cot A\}$; of O, $\{s-b, r\}$; of k, $\{-[s-a], 0\}$; of h, $\{-[s-c]\cos B, -[s-c]\sin B\}$; of f, $\{s, 0\}$; of g, $\{a+[s-b]\cos C, -[s-b]\sin C\}$; of d, $\{s\cos C-a, s\sin C\}$; of e, $\{s\cos B, s\sin B\}$.